MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
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Date: 12/01/2025 Max. Marks: 100
Time: 12.00 noon to 3.00 p.m.

Solutions and Scheme of Marking

N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. The values of a and b for which $(x-1)^2$ divides ax^4+bx^3+1 are
(A) $a=-2, b=4$ (B) $a=3, b=-4$ (C) $a=-3, b=4$ (D) $a=2, b=-3$.
Ans: (B)

2. If $f:\mathbb{R}\to\mathbb{R}$ satisfies $|f(x)-f(y)|\leq|x-y|^2$ for all $x,y\in\mathbb{R}$ and $f(1)=5$ then $f(2025)$ is
(A) 1 (B) 5 (C) 2025 (D) 0.
Ans: (B)

3. Let $z\in\mathbb{C}$. The area of triangle whose vertices are represented by $-z, iz, z-iz$ is
(A) $(1/2)|z|$ (B) $|z|$ (C) $(3/2)|z|^2$ (D) $|z|^2$.
Ans: (C)

4. The remainder when $x^{100}-2x^{51}+1$ is divided by x^2-1 is
(A) $x-2$ (B) $2x-1$ (C) $-2x+2$ (D) $x-1$.
Ans: (C)

5. If $f(x)+2f(1-x)=x^2+2$ for all real numbers x and f is differentiable function, then the value of $f'(8)$ is
(A) 0 (B) -4 (C) -3 (D) 4.
Ans: (D)

6. If the function f is periodic and for some fixed $a>0$ and for all real numbers x we have
$f(x+a)=\dfrac{1+f(x)}{1-f(x)}$, then the possible value of period is
(A) $4a$ (B) a (C) $2a$ (D) $8a$.
Ans: (A)

7. For real numbers a,b if one root of the equation $(a-b)x^2+ax+1=0$ is double the other, then the greatest value of b is
(A) 9/8 (B) 8/9 (C) 8 (D) 9.
Ans: (A)

8. The value of $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^4+n^2+1}$ is
(A) 1/3 (B) 1/4 (C) 1/5 (D) 1/2.
Ans: (D)

9. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f(r+\frac{1}{n})=f(r)$ for all rational numbers r and positive integers n. Which of the following is true?
(A) Image of f is uncountable (B) Image of f is a singleton set
(C) Image of f is two point set (D) such a function does not exist.
Ans: (B)

10. The number of regions the curves $y = x^3$ and $y = \dfrac{x^2}{x+1}$ divide the square $[0,1] \times [0,1]$ is
(A) 2 (B) 3 (C) 4 (D) 5
Ans: (C)

Part II

N.B. Each question in Part II carries 6 marks.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(f(x)) = (f(x))^2$ for all $x \in \mathbb{R}$ and $f(100) = 200$. Find all possible values of $f(500)$.
Solution: Let $f(f(x)) = (f(x))^2$. Put $x = 100$. Then we have
$f(f(100)) = f(200) = (f(100))^2 = (200)^2$. Now by the Intermediate Value Property, $[200, (200)^2] \subseteq R(f)$. [4]
As $500 \in [200, (200)^2]$, there exists a real number a such that $f(a) = 500$. Note that $f(f(a)) = f(500) = (f(a))^2 = (500)^2$. [2]

2. Let $A = \left\{\dfrac{2x+5}{3x-1} : x < 0\right\}, B = \left\{x : \dfrac{2x+5}{3x-1} < 0\right\}$. Find $\sup A, \inf A, \sup B, \inf B$ if they exist. Justify your answer.
Solution: $A = \left\{\dfrac{2x+5}{3x-1} : x < 0\right\}$
Let $f : (-\infty, 0) \to \mathbb{R}$ be a function defined as $f(x) = \dfrac{2x+5}{3x-1}$.
Then $f'(x) = \dfrac{-17}{(3x-1)^2} < 0$ and hence f is a decreasing function on $(-\infty 0)$.
Thus, $\sup(A) = \sup(f) = \lim\limits_{x \to 0} f(x) = -5$
Also, $\inf(A) = \inf(f) = \lim\limits_{x \to -\infty} f(x) = \dfrac{2}{3}$ [3]
$B = \left\{x \in \mathbb{R} : \dfrac{2x+5}{3x-1} < 0\right\}$.
Note that $x \in B \iff (2x+5 > 0$ and $3x-1 < 0)$ OR $(2x+5 < 0$ and $3x-1 > 0)$
Hence, $x \in B \iff (x > -5/2$ and $x < 1/3)$ OR $(x < -5/2$ and $x > 1/3)$
Thus, $x \in B \iff x > -5/2$ and $x < 1/3$.
So, $B = \left(-\dfrac{5}{2}, \dfrac{1}{3}\right)$ and hence, $\inf(B) = -\dfrac{5}{2}$ and $\sup(B) = \dfrac{1}{3}$. [3]

3. Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with integer coefficients such that $p(0) \neq 0$ and $p(r) = p(s) = 0$ for two integers $0 < r < s$. Prove that for some k, $a_k \leq -s$.
Solution: Since $p(s) = 0$, we have $p(x) = (x-s)q(x)$, where
$q(x) = b_0 + b_1 x + \cdots + b_{n-1}x^{n-1}, b_i$ are integers. [2]
Observe that $a_0 = -sb_0$. Thus $b_0 \neq 0$. If $b_0 > 0$, then $a_0 < -s$ and we are through. Now let $b_0 < 0$. As $q(x)$ has at least one real positive root, by Descarte's rule of sign, there is at least one sign change in the coefficients of $q(x)$. Thus there exists k such that $b_k \leq 0$ and $b_{k+1} > 0$. This implies $a_{k+1} = -sb_{k+1} + b_k \leq -s$. [4]

4. For any positive integer m, show that there exists a real $m \times m$ matrix A such that $A^3 = A + I$. Also, show that for any such A, determinant of A is positive.
Solution: Note that the matrix $A = \lambda I$ satisfies $A^3 = A + I$ if and only if $\lambda^3 = \lambda + 1$. Since $\lambda^3 - \lambda - 1$ is a cubic polynomial, it has at least one real root. Therefore such matrix A exists. [2]
Claim: The minimal polynomial of A has exactly one real root and two complex roots. Let $p(x) = x^3 - x - 1$. Since $p(1) < 0$ and $p(2) > 0$, by Intermediate Value Property, there is a root between 1 and 2. Note that the sum of the roots is zero. Therefore all roots cannot be positive. If $-1 < x < 0$, then $x^3 < 0$ but $x + 1 > 0$. Therefore $p(x)$ has no root in $(-1, 0)$. If $x < -1$, then $p'(x) = 3x^2 - 1 > 0$, and $p(-1) = -1$. Hence

$p(x) < -1$ for all $x < -1$. Therefore $p(x)$ has no root in $(-\infty, -1)$. Hence $p(x)$ has exactly one real positive root say α and two complex roots $\beta, \overline{\beta}$. If the multiplicity of α is r and of β is s, then $\det A = (\alpha)^r(\beta\overline{\beta})^s > 0$. [4]

5. A frog starts at the point $(0,0)$ in the coordinate plane and makes a sequence of jumps. In every jump frog covers a distance of 10 units and after each jump the frog is at a point whose coordinates are both integers.

   (a) Show that the frog can never reach the point $(2025, 2025)$.

   (b) Show that the frog can reach the point $(2026, 2026)$. Find the minimum number of jumps needed for the frog to achieve this.

   Solution: Note that by the given condition, the only points in $\mathbb{R}^2$ where frog can reach in one jump from $(0,0)$ are

$$(\pm 10, 0), (0, \pm 10), (\pm 6, \pm 8), (\pm 8, \pm 6).$$

   Thus, each jump corresponds to adding one of the above vectors to the previous position of the frog.

   (a) Observe that after each jump, both the coordinates of frog's position remains even. Hence, frog cannot reach to $(2025, 2025)$. [2]

   (b) Observe that $(2026, 2026) = (0,0) + 144(6,8) + 144(8,6) + (10,0) + (0,10)$
   Thus, the frog will reach to $(2026, 2026)$ by taking 290 jumps from $(0,0)$.
   Let $s = x + y$ be the sum of x and y coordinates of the frog's position (x, y) in $\mathbb{R}^2$. Thus, it starts at $s = 0$ and ends at $s = 4052$. Further, by observing the possible jumps, each jump changes the sum of x and y coordinates by atmost 14 and $4052 > 4046 = 289 \times 14$. Thus, it follows that the frog must take more than 289 jumps to reach to the destination. Hence, the minimum number of jumps needed for the frog to reach $(2026, 2026)$ from $(0,0)$ is 290. [4]

Part III

1. Let $\{s_n\}$ be a sequence of real numbers and let $\{t_n\}$ be a sequence defined by $t_k = s_{k+1} - s_k$ and $t_{k+1} - t_k = 1$ for all $k \in \mathbb{N}$.

   (a) Find s_1 if $s_8 = s_{10} = 0$. [2]

   (b) Find s_1 if $s_{20} = s_{25} = 0$. [4]

   (c) Let $s_n = s_m = 0$ for some distinct positive integers m, n. Prove that $s_k \in \mathbb{Z}$ for all $k \in \mathbb{N}$ if and only if m, n are of different parity. [6]

   Solution: Let $\{s_n\}$ be a sequence of real numbers and let $\{t_n\}$ be a sequence defined by $t_k = s_{k+1} - s_k$ and $t_{k+1} - t_k = 1$ for all $k \in \mathbb{N}$. Substituting t_k, we get
   $s_{k+2} - 2s_{k+1} + s_k = 1.$ [I]

   (a) Putting $k = 8$ in [I], we have $s_{10} - 2s_9 + s_8 = 1$. Given that $s_8 = s_{10} = 0$. Thus we get $s_9 = -1/2$. Now putting $k = 7$ in [I], we have $s_7 = 3/2$. Now by successively putting values $k = 6, 5, 4, 3, 2, 1$; we get $s_1 = 63/2$. [2]

   (b) Given that $s_{20} = s_{25} = 0$. We have $s_{21} = s_{20} + t_{20}$,
   $s_{22} = s_{21} + t_{21} = s_{21} + t_{20} + 1 = s_{20} + 2t_{20} + 1$,
   $s_{23} = s_{22} + t_{22} = s_{20} + 2t_{20} + 1 + t_{22} = s_{20} + 2t_{20} + 1 + t_{20} + 2 = s_{20} + 3t_{20} + 3$.
   Similarly, $s_{24} = s_{20} + 4t_{20} + 6$ and $s_{25} = s_{20} + 5t_{20} + 10$. Now putting $s_{20} = s_{25} = 0$, we have $t_{20} = -2$. Thus $t_{19} = -3$ and $s_{19} = s_{20} - t_{20} = 2$. Now $s_{18} = s_{19} - t_{19} = 2 + 3 = 5$. One may now use [I] and successively find values of s_k for $k < 18$. We then get $s_1 = 228$. [4]

(c) One can use the sets
$\{t_k\} = \{t_1, t_1+1, t_1+2, \cdots, t_1+(k-1), \cdots\}$
$\{s_k\} = \{s_1, s_1+t_1, s_1+2t_1+1, s_1+3t_1+1+2, \cdots\}$
$s_k = s_1 + (k-1)t_1 + (1+2+\cdots+(k-2)) = s_1 + (k-1)t_1 + \frac{(k-2)(k-1)}{2}$
Thus $s_n = s_1 + (n-1)t_1 + \frac{(n-2)(n-1)}{2}$ and $s_m = s_1 + (m-1)t_1 + \frac{(m-2)(m-1)}{2}$.
Taking difference, we get
$0 = s_n - s_m = t_1(n-m) + \frac{(n-2)(n-1)-(m-2)(m-1)}{2} = t_1(n-m) + \frac{(n-m)(n+m-3)}{2}$
Therefore, $t_1 = \frac{3-n-m}{2}$. Hence, t_1 is an integer if and only if m, n are of different parity. Now s_k is an integer if and only if t_1 is integer. [6]

2. For $0 < k \leq 1, n \in \mathbb{N}$, let $p_n(x) = x^n + x^{n-1} + \cdots + x - k$.

(a) Show that for each n, $p_n(x)$ has a unique positive real root. [2]

(b) If a_n is the positive root of $p_n(x)$, then show that the sequence $\{a_n\}$ is convergent. [5]

(c) Find $\lim_{n\to\infty} a_n$. [5]

Solution: For $0 < k \leq 1$ and for each $n \in \mathbb{N}$, we have $p_n(x) = x^n + x^{n-1} + \cdots + x - k$.

(a) Note that $p_n(0) = -k < 0$ and $p_n(k) > 0$. Thus, by intermediate value theorem, $p_n(x)$ must have a root in $(0, k)$. Further, $p_n'(x) > 0$ for all $x > 0$. Thus, $p_n(x)$ is increasing in $(0, \infty)$ and hence $p_n(x)$ has unique positive root for each n. [2]

(b) Note that $p_{n+1}(x) = p_n(x) + x^{n+1}$. Thus $P_{n+1}(a_n) = a_n^{n+1} > 0$. Thus $a_{n+1} \in (0, a_n)$ for each n. Thus, sequence (a_n) is monotonically decreasing. [5]
Further, we know that $a_n > 0 \ \forall n \in \mathbb{N}$. Thus, (a_n) is a convergent sequence.

(c) Note that $p_n(x) = \dfrac{x(1-x^n)}{1-x} - k = \dfrac{x^{n+1} - (k+1)x + k}{x-1}$
Thus, $p_n(a_n) = 0$ gives $\dfrac{a_n^{n+1}}{k+1} = a_n - \dfrac{k}{k+1}$.
Further, we have, $0 < a_n < a_2 \leq 1$, $\forall n \geq 2$. Hence, $0 < a_n^{n+1} \leq a_2^{n+1}$, $\forall n \geq 2$.
Hence the sequence (a_n^{n+1}) converges to 0 implying $\lim_{n\to\infty} a_n = \dfrac{k}{k+1}$. [5]

3. Let $L := \{(x, y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Z}\}$.

(a) Show that there is no regular hexagon with all its vertices in L. [5]

(b) Show that for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that an equilateral triangle with two of its vertices at $(0, 0)$ and $(2n, 0)$ respectively has its third vertex inside an ε-neighbourhood of a point in L. [7]

Solution: Let $L := \{(x, y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Z}\}$.

(a) If possible, let $ABCDEF$ be a regular hexagon with all its vertices in L. Since all of its vertices have integer coordinates, by Pick's Theorem we get that the area of the hexagon is given by

$$[ABCDEF] = i + \frac{b}{2} - 1$$

where i and b are the numbers of points in L in the interior and on the boundary of $ABCDEF$ respectively. This gives us that the area of hexagon $ABCDEF$ is a rational number. On the other hand, we know that for a regular hexagon, the area is also given by

$$[ABCDEF] = \frac{3\sqrt{3}}{2} AB^2$$

Since A, B are points with integer coordinates, AB^2 is also an integer. This gives us that the area of $ABCDEF$ is an irrational number, which contradicts our earlier conclusion. Hence there can be no regular hexagon with all its vertices in L. [5]

(b) Note that if an equilateral triangle has two of its vertices at $(0,0)$ and $(2n,0)$ respectively for some $n \in \mathbb{N}$, then its third vertex must be at $(n, n\sqrt{3})$. Hence the problem reduces to showing that for any $\epsilon > 0$, there exist $n, k \in \mathbb{N}$ such that $|n\sqrt{3} - k| < \epsilon$, or equivalently that, there exists $n \in \mathbb{N}$ such that $\{n\sqrt{3}\} < \epsilon$, where $\{\cdot\}$ represents the fractional part function. Let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \epsilon$. We divide the unit interval $I := [0,1)$ into N equal parts $I_1, I_2, \ldots, I_N$ of size $\frac{1}{N}$ each. Hence, we have that $I_k := [(k-1)/N, k/N)$ for each $k \in \{1, 2, \ldots, N\}$. Consider the set $M := \{n\sqrt{3} \mid n \in \mathbb{N}\}$. Since M is an infinite set, by Pigeon-Hole Principle, there exist $n_1, n_2 \in \mathbb{N}$ such that $\{n_1\sqrt{3}\}, \{n_2\sqrt{3}\} \in I_{k_0}$ for some $k_0 \in \{1, 2, \ldots, N\}$. Without loss of generality let $n_1 > n_2$. Then in this case we see that $\{(n_1 - n_2)\sqrt{3}\} \in I_1$. Since the length of I_1 is $1/N$ which is smaller than ϵ, this completes the proof. [7]

4. Let A be a square matrix of order $2k$ with entries 1 to $4k^2$ in some order exactly once.

(a) Show that there exists a row or column having two entries with their difference at least $2k^2$. [3]

(b) Show that there exists a row or column having two entries with their difference at least $2k^2 + k - 1$. [7]

(c) For $k = 2$, find such an A with the difference between any two entries in same row or column is at most 9. [4]

Solution:

(a) If 1 and $4k^2$ are in the same row or column, then $4k^2 - 1 \geq 2k^2$. Otherwise suppose a is in the intersection of row containing 1 and column containing $4k^2$. Then $\max\{4k^2 - a, a - 1\} > \frac{4k^2-1}{2}$. Therefore $\max\{4k^2 - a, a - 1\} \geq 2k^2$. [3]

(b) Define sets S and T as follows:
$S = \{(i,j) : a_{ij} \in \{1, 2, \cdots, k^2\}\}$ and $T = \{(i,j) : \textit{either } a_{ip} \textit{ or } a_{qj} \in \{1, 2, \cdots, k^2\}\}$.
That is T is a union of rows and columns in which there is an entry from $\{1, 2, \cdots, k^2\}$.
Case 1: Suppose that the entries in S forms a $k \times k$ submatrix of A.
Let $M = \max\{a_{ij} : (i,j) \in T\}$. As $(i,j) \in T$, either i^{th} row or j^{th} column has an entry from $\{1, 2, \cdots, k^2\}$. But, the entries in S is a $k \times k$ submatrix. So either i^{th} row or j^{th} column has k entries from $\{1, 2, \cdots, k^2\}$. Thus some difference is at least $M - (k^2 - k + 1)$ (because largest k entries are $k^2, k^2 - 1, k^2 - 2, \cdots, k^2 - k + 1$). In this case, we now show that $M \geq 3k^2$. This will follow if we show that $|T| \geq 3k^2$. Suppose T is a union of a rows and b columns, then $|T| = 2ka + 2kb - ab$. But $|S| = k^2 \leq ab$. Therefore $|T| = 2ka + 2kb - ab = 2ka + b(2k - a) \geq 2ka + \frac{k^2}{a}(2k - a)$ $= 2k(a + \frac{k^2}{a}) - k^2 \geq 2k(2k) - k^2 = 3k^2$.
Thus some difference is at least $3k^2 - (k^2 - k + 1) = 2k^2 + k - 1$.
Case 2: Suppose that the entries in S do not form a $k \times k$ submatrix of A. In this case, we show that $|T| \geq 3k^2 + k$. Suppose $a \geq b$. As the entries in S do not not a $k \times k$ submatrix, $a \geq k + 1$. Suppose $a = k + r \geq k + 1$. As $ab \geq k^2$, we have $b \geq k - r + 1$. But $|T| = 2ka + b(2k - a) \geq 2k(k + r) + (k - r + 1)(2k - k - r) =$ $3k^2 + k + r(r - 1) \geq 3k^2 + k$.
Thus some difference is at least $3k^2 + k - k^2 = 2k^2 + k$. [7]

(c) One possible matrix is $\begin{pmatrix} 10 & 12 & 14 & 16 \\ 9 & 11 & 13 & 15 \\ 2 & 4 & 6 & 8 \\ 1 & 3 & 5 & 7 \end{pmatrix}$ [4]