MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S. P. College, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 07/01/2018 Max. Marks: 100
Time: 12.00 noon to 3.00 p.m.
N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. Suppose $\lim_{x \to -2} \dfrac{bx^2+15x+15+b}{x^2+x-2} = L$, then the values of b and L are respectively
A) $3, -1$ B) $-3, 1$ C) $3, 1$ D) $-3, -1$.

2. Let f be a function defined on the set of positive integers by $f(1) = 1$, $f(2n) = 2f(n), f(2n+1) = 4f(n)$. The number of solutions to $f(n) = 8$ equals
A) 4 B) 3 C) 2 D) 1.

3. Let g be a continuous function which is not differentiable at 0 and let $g(0) = 8$. If $f(x) = xg(x)$, then $f'(0) =$
A) 0 B) 4 C) 2 D) 8.

4. If the circles $x^2+y^2 = 1$ and $(x-a)^2+(y-b)^2 = 1$ have exactly one point in common, then the point (a, b) lies on
A) $x^2+y^2 = 1$ B) $(x-a)^2+(y-b)^2 = 1$ C) $x^2+y^2 = 2$ D) $x^2+y^2 = 4$.

5. If x and y are **non-zero** real numbers, then x^2+xy+y^2
A) is always negative B) takes the value zero for some x, y
C) is always positive D) takes both positive and negative values.

6. The set of all real numbers x such that $\big||3-x|-|x+2|\big| = 5$ is
A) $[3, \infty)$ B) $(-\infty, -2] \cup [3, \infty)$ C) $(-\infty, -2]$ D) $(-\infty, -3] \cup [2, \infty)$.

7. Let $f(x) = \sin x + \cos x$. The infimum of $f(x)$ over the interval $[0, \pi/4]$ is
A) 0 B) 1 C) $\sqrt{2}$ D) $1/\sqrt{2}$.

8. Let A be the set of points where the function $f(x) = \cos|x-5| + |x+10|^3$ is not differentiable. Then
A) $A = \{5\}$ B) $A = \{5, -10\}$ C) $A = \{-10\}$ D) $A = \phi$.

9. How many factors of $2^5 3^6 5^2$ are perfect squares?
A) 24 B) 20 C) 30 D) 36.

10. The number of distinct real roots of the equation $2x^5+8x-7 = 0$ is
A) 1 B) 2 C) 3 D) 5.

Part II

N.B. Each question in Part II carries 6 marks.

1. Let $A_1 = \{1\}, A_{n+1} = \{3k, 3k+1 : \ k \in A_n\}$ for all $n \geq 1$ and $A = \displaystyle\bigcup_{n=1}^{\infty} A_n$. Can 2017 be written as the sum of two elements of A? Is this representation unique?

2. Prove that there do not exist distinct real numbers x, y, u, v satisfying $x^2 + y^2 = u^2 + v^2$ and
$x^3 + y^3 = u^3 + v^3$ simultaneously.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that $\displaystyle\int_0^1 f(x)x^2 dx = \frac{1}{3}f(c)$ for some $c \in [0, 1]$.

4. Suppose $f(x)$ is a polynomial with real coefficients and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Let $g = f + f' + f'' + \cdots$. Prove that $g(x) \geq 0$ for all $x \in \mathbb{R}$.

5. Let S be the set of all points $(a, b) \in \mathbb{R}^2$ for which the curves $y = x^2 + 2bx + 1$ and $y = 2a(x + b)$ do not intersect. Determine the area of S.

Part III

1. Two real numbers x and y are chosen in $(0, 1)$ randomly.
   (a) Show that the probability that 2 is the closest integer to $\dfrac{y}{x}$ is $2/15$.
   (b) If P_1 is the probability that the integer closest to $\dfrac{y}{x}$ is odd and if P_2 is the probability that the integer closest to $\dfrac{y}{x}$ is even then show that $P_1 > P_2$. [12]

2. (a) If α is a root of the polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ with real coefficients, $a_n \neq 0$, then prove that $|\alpha| \leq 1 + \displaystyle\max_{0 \leq k \leq n-1} |\frac{a_k}{a_n}|$.
   (b) Let $a_0 + 10a_1 + \cdots + 10^n a_n$ be the decimal representation of a prime number such that $a_n \geq 2, n > 1$. Prove that the polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ cannot be written as a product of two non-constant polynomials with integer coefficients. [12]

3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x + \tan^{-1} x$.
   (a) Show that f is bijection.
   (b) Define $h : \mathbb{R} \to \mathbb{R}$ by
   $$h(x) = \begin{cases} f(x) & \text{if } \ x \geq c, \\ f^{-1}(x) & \text{if } \ x < c. \end{cases}$$
   Find all real numbers c such that h is continuous at c.
   (c) Find all real numbers c such that h is differentiable at c. [13]

4. Let A, B be $n \times n$ $(n \geq 2)$ nonsingular matrices with real entries.
   (a) If $A^{-1} + B^{-1} = (A + B)^{-1}$, then prove that $\det A = \det B$.
   (b) Find examples of matrices A, B satisfying $A^{-1} + B^{-1} = (A + B)^{-1}$.
   (c) Find examples of matrices A, B with complex entries such that
   $A^{-1} + B^{-1} = (A + B)^{-1}$, but $\det A \neq \det B$. [13]