MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S. P. College, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 06/01/2019 Max. Marks: 100
Time: 12.00 noon to 3.00 p.m.
N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. The values of k for which the line $y = kx$ intersects the parabola $y = (x-1)^2$ are
A) $k \leq 0$ B) $k \geq -4$ C) $k \geq 0$ or $k \leq -4$ D) $-4 \leq k \leq 0$.

2. Let $M_2(\mathbb{Z}_2)$ denote the set of all 2×2 matrices with entries from $\mathbb{Z}_2$, where $\mathbb{Z}_2$ denotes the set of integers modulo 2. The function $f : M_2(\mathbb{Z}_2) \to M_2(\mathbb{Z}_2)$ given by $f(x) = x^2$ is
A) injective but not surjective B) bijective
C) surjective but not injective D) neither injective nor surjective.

3. Consider the sequence $4, 0, 4.1, 0, 4.11, 0, 4.111, 0, \cdots$. This sequence
A) converges to $4\frac{1}{9}$ B) has no convergent subsequence
C) is unbounded D) is not convergent and has supremum $4\frac{1}{9}$.

4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = (x-2)|(x-2)(x-3)|$. The function f is
A) differentiable at $x = 2$ but not at $x = 3$ B) differentiable at $x = 3$ but not at $x = 2$
C) differentiable at $x = 2$ and $x = 3$ D) neither differentiable at $x = 2$ nor at $x = 3$.

5. The equation $z^2 + \bar{z}^2 = 2$ represents the
A) parabola B) pair of lines C) hyperbola D) ellipse.

6. The differential equation of the family of parabolas having their vertices at the origin and their foci on the X-axis is
A) $2xdy - ydx = 0$ B) $xdy + ydx = 0$ C) $2ydx - xdy = 0$ D) $dy - xdx = 0$.

7. The number of solutions of the equation $\sqrt{1 - \sin x} = \cos x$ in $[0, 5\pi]$ is equal to
A) 3 B) 6 C) 8 D) 11.

8. Consider $\triangle ABC$. Take 3 points on AB, 4 on BC and 5 on CA such that none of the points are vertices of $\triangle ABC$. The number of triangles that can be constructed using these points is
A) 60 B) 205 C) 145 D) 120.

9. The number of primes p such that $p, p+10, p+14$ are all prime numbers is
A) 0 B) 1 C) 3 D) infinitely many.

10. A relation R is defined on the set of positive integers as xRy if $2x + y \leq 5$. The relation R is
A) reflexive B) symmetric C) transitive D) None of these.

Part II

N.B. Each question in Part II carries 6 marks.

1. Find all polynomials $p(x)$ such that $(p(x))^2 = 1 + xp(x+1)$ for all real numbers x.

2. A transposition of a vector X of length n is created by switching exactly two distinct entries of a vector X. For example, $(1,3,2,4)$ is a transposition of the vector $(1,2,3,4)$ of length 4. Find a vector X if it is given that all the vectors below are transpositions of X: $S = (0,1,1,1,0,0,0,1), T = (1,0,1,1,1,0,0,0), U = (1,0,1,0,1,0,0,1)$, $V = (1,1,1,1,0,0,0,0), W = (1,0,0,1,0,0,1,1)$.

3. In the complex plane, let u, v be two distinct solutions of $z^{2019} - 1 = 0$. Find the probability that $|u+v| \geq 1$.

4. Let $f : [a,b] \to [a,b]$ be a continuous function which is differentiable on (a,b) and $f(a) = a, f(b) = b$. Prove that there exist two distinct points x_1 and x_2 in (a,b) such that $f'(x_1)f'(x_2) = 1$.

5. Prove that there do not exist functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x^{2018}$ and $g(f(x)) = x^{2019}$.

Part III

1. Let $f(x) = a_0x^n + \cdots + a_n$ be a non-constant polynomial with real coefficients satisfying
$$f(x)f(2x^2) = f(2x^3 + x)$$
for all real numbers x.
   (a) Show that $a_n \neq 0$.
   (b) Show that f has no real root.
   (c) Find a polynomial f satisfying $f(x)f(2x^2) = f(2x^3 + x)$ for all real numbers x.

   [13]

2. For a subset $X = \{x_1, x_2, \cdots, x_n\}$ of the set of positive integers, $X + X$ denotes the set $\{x_i + x_j \ : \ i \neq j\}$ and $|X|$ denotes the number of elements in X.
   (a) Find subsets A, B of positive integers such that $|A| = |B| = 4, A \neq B$ and $A + A = B + B$.
   (b) Do there exist subsets A, B of positive integers such that $|A| = |B| = 3, A \neq B$ and $A + A = B + B$?
   (c) Show that if $n = 2^k$, then there exist subsets A, B of positive integers such that $|A| = |B| = n, A \neq B$ and $A + A = B + B$. [13]

3. On the real line place an object at 1. After every flip of a fair coin, move the object to the right by 1 unit if the outcome is Head and to the left by 1 unit if the outcome is Tail. Let N be a fixed positive integer. Game ends when the object reaches either 0 or N. Let $P(N)$ denote the probability of the object reaching N.
   (a) Find $P(3)$.
   (b) Find the formula for $P(N)$ for any positive integer N. [12]

4. Let f be a real valued differentiable function on $(0, \infty)$ satisfying
   (a) $|f(x)| \leq 5$ and
   (b) $f(x)f'(x) \geq \sin x$ for all $x \in (0, \infty)$.

   Does $\lim_{x \to \infty} f(x)$ exist? [12]
