MADHAVA MATHEMATICS COMPETITION (Second Round)
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S. P. College, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: /02/2022 Max. Marks: 50
Time: 12.00 noon to 2.00 p.m.
N.B.: Part I carries 30 marks and Part II carries 20 marks.

Part I

N.B. Each question in Part I carries 6 marks.

1. Let the positive integers a, b, c be such that $a \geq b \geq c$ and $(a^x - b^x - c^x)(x-2) > 0$ for all $x \neq 2$. Show that a, b, c are sides of a right angled triangle.
Solution: Let $f(x) = (a^x - b^x - c^x)$ where $x \in \mathbb{R}$.
For $x < 2$ implies $f(x) < 0$
For $x > 2$ implies $f(x) > 0$
By continuity of $f(x)$ we have
$\lim_{x \to 2} f(x) = f(2) = 0$
Thus $a^2 = b^2 + c^2$

2. Find all real numbers x, y such that the fractional part of $\dfrac{x+4y+1}{x^2+y^2+19}$ is $\dfrac{1}{2}$.

Solution: Note that since $x^2+y^2+19 > 0$ and $\{\dfrac{x+4y+1}{x^2+y^2+19}\}$ is $\dfrac{1}{2}$, we have $x+4y+1 > 0$.
We can prove that $(x+4y+1) < (x^2+y^2+19)$. Since $x^2 - x + y^2 - 4y + 18 = (x-1/2)^2 + (y-4)^2 + 7/4 > 0$, we have
$\{\dfrac{x+4y+1}{x^2+y^2+19}\} = \dfrac{x+4y+1}{x^2+y^2+19} = \dfrac{1}{2}$
Thus $2x + 8y + 2 = x^2 + y^2 + 19$
i.e. $(x-1)^2 + (y-4)^2 = 0$. We get $(x, y) = (1, 4)$.

3. Let f be a quadratic polynomial. Show that there exist quadratic polynomials g, h such that $f(x)f(x+1) = g(h(x))$. **Solution:**
$f(x) = a(x-\alpha)(x-\beta)$
$f(x+1) = a(x-\alpha+1)(x-\beta+1)$
$f(x) * f(x+1) = a^2(x^2 - x(\alpha+\beta-1) + \alpha\beta - \alpha) * (x^2 - x(\alpha+\beta-1) + \alpha\beta - \beta)$
Define $g(x) = a^2(x-\alpha)(x-\beta)$
$h(x) = x^2 - x(\alpha+\beta-1) + \alpha\beta$

4. Determine the number of all $m \times n$ matrices with entries 0 or 1 such that the number of 1's in each row and the number of 1's in each column are all even.
Solution:
Fill $m-1$ by $n-1$ submatrix in any way. The number of ways to do this is $2^{(m-1)(n-1)}$. Since only 0 and 1 is allowed as entry and we have to just take care of parity, the remaining entries in the last row and column are uniquely forced. Thus the number of required $m \times n$ matrices is $2^{(m-1)(n-1)}$.

5. Find all non-negative integer solutions to the system of equations

$$3x^2 - 2y^2 - 4z^2 + 54 = 0$$
$$5x^2 - 3y^2 - 7z^2 + 74 = 0$$

Solution: Multiplying first equation by 5 and second by 3 we get

$$15x^2 - 10y^2 - 20z^2 + 5 * 54 = 0$$
$$15x^2 - 9y^2 - 21z^2 + 74 * 3 = 0$$

Substracting the second equation from the first, we get

$$z^2 - y^2 = -48$$
$$(y - z)(y + z) = 48$$

Trying the pairs 2×24, 6×8, 4×12
$y - z = 2, y + z = 24$, gives $y = 13, z = 11$ gives $x = 16$
$y - z = 6, y + z = 8$, gives $y = 7, z = 1$ gives $x = 4$
$y - z = 4, y + z = 12$ gives $y = 8, z = 4$ does not give a valid x.

Part II

N.B. Each question in Part II carries 10 marks.

1. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function such that f' is continuous and $f(0) = 0, f(1) = 1$.

   (a) Show that there exists x_1 in $(0,1)$ such that $\dfrac{1}{f'(x_1)} = 1$. [1]

   (b) Show that there exist distinct x_1, x_2 in $(0,1)$ such that $\dfrac{1}{f'(x_1)} + \dfrac{1}{f'(x_2)} = 2$. [4]

   (c) Show that for a positive integer n, there exist distinct $x_1, x_2, \cdots, x_n$ in $(0,1)$ such that $\displaystyle\sum_{i=1}^{n} \frac{1}{f'(x_i)} = n$. [5]

Solution:

Since $f(0) = 0, f(1) = 1$ we have $\dfrac{f(1) - f(0)}{1 - 0} = f'(c)$

Therefore result is true for $n = 1$.

If $f'(x) \geq 1 \forall x$, then

$\dfrac{f(x)}{x} = f'(c_1) \geq 1$

Therefore $f(x) \geq x$.

If $f(x_0) > x_0$, then $1 - f(x_0) < 1 - x_0$

$f'(c_2) = \dfrac{1 - f(x_0)}{1 - x_0} < 1$ which is a contradiction. Thus $f(x) = x$.

(≤)

If $f'(x) \geq 1$ $\forall x$ then $f(x) = x$

There exists $a, b \in \mathbb{R}$ such that $f'(a) < 1$ and $f'(b) > 1$

Therefore $[1 - \epsilon, 1 + \epsilon] \subseteq Ran(f')$. Therefore we can find x_1, such that

$f'(x_1) = 1 - \epsilon/2$ i.e. $\dfrac{1}{f'(x_1)} = \dfrac{2}{2 - \epsilon}$.

Now we want to find x_2 such that $f'(x_2)$ satisfies

$\dfrac{1}{f'(x_2)} = 2 - \dfrac{1}{f'(x_1)} = 2 - \dfrac{2}{2 - \epsilon} = \dfrac{2 - 2\epsilon}{2 - \epsilon}$ i.e. $f'(x_2) = \frac{2-\epsilon}{2-2\epsilon}$. This is definitely greater than 1.

If $\dfrac{2 - \epsilon}{2 - 2\epsilon} < 1 + \epsilon$; then we can find such x_2.

Now $(1 + \epsilon)(2 - 2\epsilon) = 2 - 2\epsilon^2$.

Assuming without loss of generality that $\epsilon < 1$;

$2 - \epsilon < 2 - 2\epsilon^2$

Therefore $1 < \dfrac{2 - \epsilon}{2 - 2\epsilon} < 1 + \epsilon$

i.e. $\dfrac{2 - \epsilon}{2 - 2\epsilon} \in \text{Range}(f')$

Therefore it is possible to select x_2 such that

$\dfrac{1}{f'(x_1)} + \dfrac{1}{f'(x_2)} = 2$

Now $\dfrac{1}{f'(x_1)} < 1 < \dfrac{1}{f'(x_2)}$

Therefore $\exists\ \epsilon'$ such that

$$\frac{1}{f'(x_1)} < 1-\epsilon' \quad 1+\epsilon' < \frac{1}{f'(x_2)},$$
i.e. in particular $[1-\epsilon', 1+\epsilon'] \subseteq \text{Range}(f)$

By similar argument $\exists x_3$ and x_4 such that $\frac{1}{f'(x_3)} + \frac{1}{f'(x_4)} = 2$

Therefore $\frac{1}{f'(x_1)} + \frac{1}{f'(x_2)} + \frac{1}{f'(x_3)} + \frac{1}{f'(x_4)} = 4$ and by contradiction x_1, x_2, x_3, x_4 are distinct. We can continue this process and get $x_1, x_2, x_3, \ldots, x_{2n}$ all distinct such that $\sum_{i=1}^{2k} \frac{1}{f'(x_i)} = 2k$. We have already seen that there exists $c = x$ such that $f'(x) = 1$. Since $f'(x_i) \neq 1 \ \forall\ i = 1, 2, \ldots, 2k$; this c is different from all x_i.

Thus $\exists$ 2k+1 points such that $\sum_{i=1}^{2k+1} \frac{1}{f'(x_i)} = 2k+1$.

2. Let $\mathcal{P}_n$ denote the collection of polynomials of degree n such that the polynomial and all its derivatives have integer roots.

   (a) Find a polynomial in $\mathcal{P}_2$ having at least two distinct roots. [2]

   (b) Find a polynomial in $\mathcal{P}_3$ having at least two distinct roots. [3]

   (c) For any polynomial P in $\mathcal{P}_n$, show that the arithmetic mean of all roots of P is also an integer. [5]

Solution:

i) Let $f = (x-\alpha)(x-\beta)$; then $f' = (x-\alpha) + (x-\beta) = 2x - (\alpha+\beta)$

Thus the condition is $\alpha+\beta$ must be even.

ii) Let $f = (x-\alpha)(x-\beta)(x-\gamma)$

Then $f' = (x-\alpha)(x-\beta) + (x-\alpha)(x-\gamma) + (x-\beta)(x-\gamma)$

Let $f = (x-\alpha)^2(x-\gamma)$. Then $f' = (x-\alpha)^2 + 2(x-\alpha)(x-\beta)$

$f'' = 2(x-\alpha) + 2(x-\alpha) + 2(x-\beta) = 4(x-\alpha) + 2(x-\beta)$

$= 6x - 4\alpha - 2\beta$

$f'' = 0$ if $x = \frac{4\alpha+2\beta}{6}$

So $6|4\alpha+2\beta$. Put $\beta = 1 and \alpha = 4$. We can take $f(x) = (x-4)^2(x-1)$.

iii) Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \in P_n$. Let the roots of $p(x)$ be $\alpha_1, \alpha_2, \ldots, \alpha_n$

Now $p'(x) = n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + a_1$

$p''(x) = n(n-1)a_n x^{n-2} + (n-1)(n-2)a_{n-1}x^{n-3} + \ldots + 2a_2$ and so on.

$p^{(n-1)}(x) = n! a_n x + (n-1)! a_{n-1}$

Since $p^{(n-1)}(x)$ has integer roots, it is equal to $\frac{-(n-1)! a_{n-1}}{n! a_n}$

$\frac{\sum_{i=1}^{n} \alpha_i}{n} = \frac{-(n-1)a_{n-1}}{n a_n}$ is thus an integer.