MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
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Date: 12/01/2025 Max. Marks: 100
Time: 12.00 noon to 3.00 p.m.
N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. The values of a and b for which $(x-1)^2$ divides ax^4+bx^3+1 are
   (A) $a=-2, b=4$ (B) $a=3, b=-4$ (C) $a=-3, b=4$ (D) $a=2, b=-3$.

2. If $f:\mathbb{R}\to\mathbb{R}$ satisfies $|f(x)-f(y)|\leq|x-y|^2$ for all $x,y\in\mathbb{R}$ and $f(1)=5$ then $f(2025)$ is
   (A) 1 (B) 5 (C) 2025 (D) 0.

3. Let $z\in\mathbb{C}$. The area of triangle whose vertices are represented by $-z, iz, z-iz$ is
   (A) $(1/2)|z|$ (B) $|z|$ (C) $(3/2)|z|^2$ (D) $|z|^2$.

4. The remainder when $x^{100}-2x^{51}+1$ is divided by x^2-1 is
   (A) $x-2$ (B) $2x-1$ (C) $-2x+2$ (D) $x-1$.

5. If $f(x)+2f(1-x)=x^2+2$ for all real numbers x and f is differentiable function, then the value of $f'(8)$ is
   (A) 0 (B) -4 (C) -3 (D) 4.

6. If the function f is periodic and for some fixed $a>0$ and for all real numbers x we have
   $f(x+a)=\dfrac{1+f(x)}{1-f(x)}$, then the possible value of period is
   (A) $4a$ (B) a (C) $2a$ (D) $8a$.

7. For real numbers a, b if one root of the equation $(a-b)x^2+ax+1=0$ is double the other, then the greatest value of b is
   (A) 9/8 (B) 8/9 (C) 8 (D) 9.

8. The value of $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^4+n^2+1}$ is
   (A) 1/3 (B) 1/4 (C) 1/5 (D) 1/2.

9. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f(r+\frac{1}{n})=f(r)$ for all rational numbers r and positive integers n. Which of the following is true?
   (A) Image of f is uncountable (B) Image of f is a singleton set
   (C) Image of f is two point set (D) such a function does not exist.

10. The number of regions the curves $y=x^3$ and $y=\dfrac{x^2}{x+1}$ divide the square $[0,1]\times[0,1]$ is
    (A) 2 (B) 3 (C) 4 (D) 5

Part II

N.B. Each question in Part II carries 6 marks.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(f(x)) = (f(x))^2$ for all $x \in \mathbb{R}$ and $f(100) = 200$. Find all possible values of $f(500)$.

2. Let $A = \left\{ \dfrac{2x+5}{3x-1} : x < 0 \right\}, B = \left\{ x : \dfrac{2x+5}{3x-1} < 0 \right\}$. Find $\sup A, \inf A, \sup B, \inf B$ if they exist. Justify your answer.

3. Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with integer coefficients such that $p(0) \neq 0$ and $p(r) = p(s) = 0$ for two integers $0 < r < s$. Prove that for some k, $a_k \leq -s$.

4. For any positive integer m, show that there exists a real $m \times m$ matrix A such that $A^3 = A + I$. Also, show that for any such A, determinant of A is positive.

5. A frog starts at the point $(0, 0)$ in the coordinate plane and makes a sequence of jumps. In every jump frog covers a distance of 10 units and after each jump the frog is at a point whose coordinates are both integers.

   (a) Show that the frog can never reach the point $(2025, 2025)$.

   (b) Show that the frog can reach the point $(2026, 2026)$. Find the minimum number of jumps needed for the frog to achieve this.

Part III

1. Let $\{s_n\}$ be a sequence of real numbers and let $\{t_n\}$ be a sequence defined by $t_k = s_{k+1} - s_k$ and $t_{k+1} - t_k = 1$ for all $k \in \mathbb{N}$.

   (a) Find s_1 if $s_8 = s_{10} = 0$. [2]

   (b) Find s_1 if $s_{20} = s_{25} = 0$. [4]

   (c) Let $s_n = s_m = 0$ for some distinct positive integers m, n. Prove that $s_k \in \mathbb{Z}$ for all $k \in \mathbb{N}$ if and only if m, n are of different parity. [6]

2. For $0 < k \leq 1, n \in \mathbb{N}$, let $p_n(x) = x^n + x^{n-1} + \cdots + x - k$.

   (a) Show that for each n, $p_n(x)$ has a unique positive real root. [2]

   (b) If a_n is the positive root of $p_n(x)$, then show that the sequence $\{a_n\}$ is convergent. [5]

   (c) Find $\lim\limits_{n \to \infty} a_n$. [5]

3. Let $L := \{(x, y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Z}\}$.

   (a) Show that there is no regular hexagon with all its vertices in L. [5]

   (b) Show that for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that an equilateral triangle with two of its vertices at $(0, 0)$ and $(2n, 0)$ respectively has its third vertex inside an ε-neighbourhood of a point in L. [7]

4. Let A be a square matrix of order $2k$ with entries 1 to $4k^2$ in some order exactly once.

   (a) Show that there exists a row or a column having two entries with their difference at least $2k^2$. [3]

   (b) Show that there exists a row or a column having two entries with their difference at least $2k^2 + k - 1$. [7]

   (c) For $k = 2$, find such an A with the difference between any two entries in same row or column is at most 9. [4]