MADHAVA MATHEMATICS COMPETITION

Organized by
Department of Mathematics, S. P. College, Pune
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Date : 3 / 1 / 2010 **Marks : 100**

Time : 12.00 noon to 3.00 p.m.

N. B.

1) All questions are compulsory.

2) Part I carries 20 marks, Part II carries 20 marks and Part III carries 60 marks.

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Part I

N.B. Each question in part I carries 2 marks.

1. The number of subsets of the set $\{1, 2, \cdots, 10\}$ containing at least one odd integer is
(a) 2^{10} (b) 2^5 (c) ${}^{10}C_5$ (d) $2^{10} - 2^5$.

2. $1^2 - 2^2 + 3^2 - 4^2 + ... + (2009)^2 - (2010)^2$ is equal to
(a) zero
(b) -2021055
(c) -2019045
(d) -1010555.

3. The coefficient of $(x-1)^3$ in the Taylor expansion of $(x-1)^3\cos(\pi x)$ about $x = 1$ is
(a) -1 (b) 1 (c) 6 (d) -6.

4. The number of non-zero solutions of $z^2 + 2\bar{z} = 0$ is
(a) 2 (b) 3 (c) 4 (d) 5.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by
$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 5x - 6 & \text{if } x \notin \mathbb{Q}. \end{cases}$$
Then f is continuous at
(a) no real number (b) -2 and -3 (c) all rationals (d) 2 and 3.

6. The number of negative solutions of the equation $e^x - \sin x = 0$ is
(a) 1 (b) 2 (c) 0 (d) infinite.

7. Let $a_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n$ and $b_n = (2+\sqrt{3})^n - (2-\sqrt{3})^n$, and let T_n denote the area of the triangle with sides $a_n - 1, a_n, a_n + 1$. Then
(a) a_n, b_n, T_n are all integers for each $n \in \mathbb{N}$.
(b) a_n, T_n are all integers for each $n \in \mathbb{N}$.
(c) T_n is not an integer for each $n \in \mathbb{N}$.
(d) a_n is an integer for even n and b_n is an integer for odd n.

8. Let $A = (a_{ij})$ be an $m \times n$ matrix where
$$a_{ij} = \begin{cases} 0 & \text{if } i + j \text{ is even} \\ 1 & \text{if } i + j \text{ is odd.} \end{cases}$$
Then the rank of $A = (a_{ij})$ is
(a) m (b) n (c) 2 (d) 3.

9. The last two digits of 17^{400} are
(a) 17 (b) 09 (c) 01 (d) 89.

10. The number of values of a for which the equation $x^3 - x + a = 0$ has a double root is
(a) 0 (b) 1 (c) 2 (d) infinite.

Part II

N.B. Each question in part II carries 5 marks.

1. Find the remainder when $f(x^3)$ is divided by $f(x)$ where $f(x) = 1 + x + x^2$.

2. Let $f : [0, 1] \to [0, 1]$ be a function defined as follows: $f(1) = 1$, and if $a = 0.a_1a_2a_3....$ is the decimal representation of a (which does not end with a chain of 9's), then $f(a) = 0.0a_10a_20a_3.....$ Discuss the continuity of f at 0.392.

3. A bubble chamber contains 3 types of particles. 100 of type x, 200 of type y and 300 of type z. Whenever x and y particles collide they both become z particles, likewise when y and z collide they both become x, and when x and z collide they both become y.
   Can the particles in the chamber evolve so that there remain particles of only one type?

4. Let $f, g : \mathbb{N} \to \mathbb{N}$ be functions such that f is onto, g is one-one and $f(n) \geq g(n)$ for all $n \in \mathbb{N}$. Prove that $f = g$.

Part III

N.B. Each question in part III carries 12 marks.

1. Abhi and Ash play the following game:
   A blank 2010×2010 array is taken. Abhi starts the game by writing a real number in any one of the squares of the array. Then Ash writes a real number in any blank square of the array. The game is continued till all the squares are filled with numbers. Abhi wins the game if the determinant of the resulting matrix is non-zero and Ash wins the game if the determinant of the resulting matrix is 0.
   Show that Ash can always win the game.

2. Show that the polynomial equation with real coefficients $a_nx^n + a_{n-1}x^{n-1} + ... + a_3x^3 + x^2 + x + 1 = 0$ cannot have all real roots.

3. Find the sum $\sum_{j=0}^{n} \sum_{i=j}^{n} {}^nC_i \, {}^iC_j$.

4. Find the g.c.d. of the numbers $\{2^{13} - 2, 3^{13} - 3, 4^{13} - 4, \cdots, 13^{13} - 13\}$.

5. Let $\{a_n\}$ be a sequence of real numbers. Suppose $\{a_{sn}\}$ converges for every fixed positive integer $s > 1$.
   1) If $a_{sn} \to a$ and $a_{tn} \to b$ for some fixed positive integers s and t, then is $a = b$? Justify.
   2) Is the sequence $\{a_n\}$ convergent? Justify.