MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Bhaskaracharya Pratishthana, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai
Solutions and Scheme of Marking

Date: 07/01/2024 Max. Marks: 100
Time: 12.00 noon to 3.00 p.m.
N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. Find constants a, b such that $\lim_{x \to 0} \frac{\sqrt{ax+b}-2}{x} = 1$.
   (A) $a=-4, b=-4$ (B) $a=4, b=4$ (C) $a=-3, b=4$ (D) $a=4, b=-2$.
   Ans: (B)

2. The value of n for which $i+2(i)^2+3(i)^3+4(i)^4+\cdots+n(i)^n$ equals $-16+15i$ is
   (A) 20 (B) 25 (C) 30 (D) 15.
   Ans: (C)

3. Find the value of a real number b for which the sum of the squares of the zeros of $x^2-(b-2)x-b-1$ is minimal.
   (A) 2 (B) 3 (C) -5 (D) 1.
   Ans: (D)

4. Let S be the set of all last two digits of the powers of 3. (For example, $03, 09, 27, 81, 43 \in S$) Then, the number of distinct elements of S is
   (A) 20 (B) 25 (C) 30 (D) 40.
   Ans: (A)

5. The sum of the roots of $x^2-31x+220 = 2^x(31-2x-2^x)$ is
   (A) 10 (B) 7 (C) 3 (D) 4.
   Ans: (B)

6. The number of regions in which the plane gets divided by curves $(\sin t, \sin 2t)$ and $(\cos t, \cos 2t)$ for $t \in \mathbb{R}$ is
   (A) 5 (B) 7 (C) 6 (D) 4.
   Ans: (C)

7. In how many ways 6 persons can exchange seats among them in a row such that no one occupies his seat in a original position and exactly two of them have mutual exchange?
   (A) $\binom{6}{2} \times 6$ (B) $\binom{6}{2} \times 4!$ (C) $\binom{6}{2} \times 3$ (D) $\binom{6}{2} \times 4$.
   Ans: (A)

8. Let $A = \{x \in (0,3) \mid [x]^2 = [x^2]\}$, where $[x]$ denotes the greatest integer less than or equal to x. Let $M = \sup A$. Then
   (A) $M \in A, M \notin (0,3)$ (B) $M \notin A, M \notin (0,3)$
   (C) $M \in A, M \in (0,3)$ (D) $M \notin A, M \in (0,3)$.
   Ans: (D)

9. If $4047 = x + \frac{x}{1+2} + \frac{x}{1+2+3} + \frac{x}{1+2+3+4} + \cdots + \frac{x}{1+2+3+\cdots+4047}$, then $x =$

(A) 2000 (B) 2024 (C) 2002 (D) 2004.
Ans: (B)

10. Let $A(0,0), B(0,23), C(23,0)$ be the points in the plane. The number of points with integral coordinates that lie inside the triangle ABC (not on the boundary) is
(A) 253 (B) 242 (C) 231 (D) 219.
Ans: (C)

Part II

N.B. Each question in Part II carries 6 marks.

1. Following operations are permitted with a quadratic $ax^2 + bx + c$

   (i) Switch a and c

   (ii) Replace x by $x + t$ for $t \in \mathbb{R}$.

   Can you convert $x^2 - x - 2$ into $x^2 - x - 1$ with repeated operations (i) and (ii)?
   Solution: Let $f(x) = ax^2 + bx + c$ and let $g(x)$ be the polynomial obtained after performing both the above operations (i) and (ii) for some $t \in \mathbb{R}$. Then, $g(x) = cx^2 + (2tc + b)x + ct^2 + bt + a$.
   We observe that the discriminant of $f(x)$ and $g(x)$ is same. That is, the discriminant is invariant under the operations (i), (ii). [4]
   Now, the discriminant of $x^2 - x - 2$ is 9 whereas that of $x^2 - x - 1$ is 5. Thus, it is not possible to convert $x^2 - x - 2$ into $x^2 - x - 1$ with repeated operations (i) and (ii). [2]

2. Consider a right angled triangle PRQ with coordinates of the vertices integers. If slope and length of the hypotenuse PQ are integers, then show that PQ is parallel to the X-axis.
   Solution: Let $P(m_1, n_1)$ and $Q(m_2, n_2)$ be vertices with integer coordinates. The slope $\lambda = \dfrac{n_2 - n_1}{m_2 - m_1}$ is also integer. Suppose the length of the hypotenuse PQ is m which is also integer. Then $m^2 = (m_2 - m_1)^2 + (n_2 - n_1)^2 = (m_2 - m_1)^2 + \lambda^2(m_2 - m_1)^2 = (1 + \lambda^2)(m_2 - m_1)^2$. [4]
   Thus $(1 + \lambda^2)$ is a square implying that $\lambda = 0$. Therefore $n_1 = n_2$. Thus PQ is parallel to the X-axis. [2]

3. Let $1 \le a_1 < a_2 < \cdots < a_\ell \le n$ be integers with $\ell > \dfrac{n+1}{2}$. Show that there exist i, j, k with $1 \le i < j < k \le \ell$ such that $a_i + a_j = a_k$.

   The statement above is not true if n is an even integer.
   For example, if $n = 4$, choose $l = 3$ and consider the set $\{2, 3, 4\}$. Then the property claimed in the problem does not hold. In general, if $n = 2k$, choose $l = \lfloor \frac{2k+1}{2} \rfloor + 1$ and then the choice of elements $\{n - l + 1, n - l + 2 \cdots, n - 1, n\}$ gives a contradiction.
   The statement of the problem is correct if n is an odd integer:
   For an odd integer $n = 2k + 1$, let $\ell \ge \frac{n+1}{2} + 1$.
   Let $A = \{a_2, \cdots, a_\ell\}$ and $B = \{a_2 - a_1, a_3 - a_1, \cdots, a_\ell - a_1\}$.
   Then A and B are subsets of first n natural numbers with $|A| + |B| = 2\ell - 2 \ge n + 1$
   Therefore $A \cap B$ is non-empty. Hence there exist a_i, a_j such that $i \ge 2$ and $i \ne j$ such that $a_i = a_j - a_1$.

   Note :
   1. As there is an error in the problem, please give 6 marks to all the students.
   2. As per every year, top few papers will be centrally moderated. We shall re-assess the solutions of this problem at the time of moderation, and due credit to the correct answers/partial answers/counterexamples will be given at that time.

4. The sequence (a_n) is defined by

$$a_1 = 0, |a_n| = |a_{n-1} + 1| \text{ for each } n \geq 2.$$

For every positive integer n, prove that, $\dfrac{a_1 + a_2 + \cdots + a_n}{n} \geq -\dfrac{1}{2}$.
When does the equality hold?

Solution 1: For each n, the number $\dfrac{a_1 + a_2 + \cdots + a_n}{n}$ is minimum if a_n is chosen to be a non-positive value of $(a_{n-1} + 1)$. [4]
Thus $\dfrac{a_1 + a_2 + \cdots + a_n}{n}$ is minimized if $a_1 = 0, a_2 = -1, a_3 = 0, a_4 = -1, \cdots$ [2]

Solution 2: Squaring each term, we get $a_1^2 = 0, a_2^2 = (a_1 + 1)^2, a_3^2 = (a_2 + 1)^2, \cdots,$
$a_{n+1}^2 = (a_n + 1)^2$. Adding all these terms, we get
$a_1^2 + a_2^2 + a_3^2 + \cdots + a_{n+1}^2 = (a_1 + 1)^2 + (a_2 + 1)^2 + \cdots + (a_n + 1)^2$
$= a_1^2 + a_2^2 + \cdots + a_n^2 + 2(a_1 + a_2 + \cdots + a_n) + n.$
This implies that $a_{n+1}^2 = 2(a_1 + a_2 + \cdots + a_n) + n \geq 0$.

Hence $\dfrac{a_1 + a_2 + \cdots + a_n}{n} \geq -\dfrac{1}{2}$. [4]
Equality holds if and only if n is even and the sequence is $\{0, -1, 0, -1. \cdots\}$ [2]

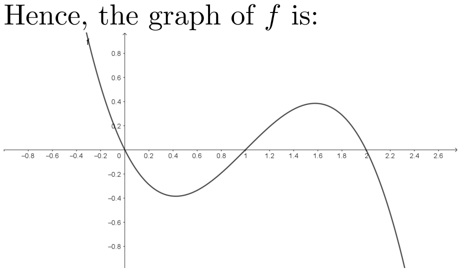
5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x(x-1)(2-x)$.

$$\text{Let } S = \{x \in \mathbb{R} : f(x+t) > f(x) \text{ for some } t > 0\}.$$

(a) Draw the graph of f.
(b) Is the set S non-empty? Justify.
(c) Is the set S bounded? Find $\sup S$, if it exists.

Solution:

(a) Observe: f is a polynomial of degree 3. Further, $f(x) > 0$ if $x < 0$ or $1 < x < 2$ and $f(x) < 0$ if $0 < x < 1$ or $x > 2$.
Hence, the graph of f is:



[1]

(b) $S \neq \emptyset$. Since, $f(1.1) = 1.1 \times 0.1 \times 0.9 > 0 = f(1)$, hence, $1 \in S$. [1]

(c) $f(x) = -x^3 + 3x^2 - 2x$. Thus, $f'(x) = -3x^2 + 6x - 2 = -3(x-1)^2 + 1$
For $x \geq 2$, $f'(x) \leq -2 < 0$ and hence, f is decreasing on $[2, \infty)$. Thus, $x \notin S$, $\forall x \geq 2$.
Hence, S is bounded above by 2.
$f'(x) = 0 \implies x = 1 \pm \dfrac{1}{\sqrt{3}}$. Let $x_1 = 1 - \dfrac{1}{\sqrt{3}}$ and $x_2 = 1 + \dfrac{1}{\sqrt{3}}$.
$f''(x) = -6x + 6$. $f''(x_1) > 0$ and $f''(x_2) < 0$, thus by Second derivative test, f has local minimum at x_1 and local maximum at x_2.
It can be seen that f is strictly decreasing on $(-\infty, x_1)$ and on (x_2, ∞) and f is strictly increasing on (x_1, x_2). Now, for any $x \leq 0$, we get, $f'(x) < -2$.
Let $n \in \mathbb{Z}$ be such that $n < n + 1 \leq 0$. Then, by Lagrange's mean value theorem, there exist $c \in (n, n+1)$ such that $f(n+1) - f(n) = f'(c) < -2$, as $c < 0$.

Hence, we get, $f(n) > f(n+1) + 2$, for all $n < n+1 \leq 0$.
Thus, as $n \to -\infty$, $f(n) \to \infty$.
$\Longrightarrow$ there exist $n_0 \in \mathbb{Z}$ with $n_0 < 0$ such that $f(x) > f(x_2)$ for all $x \leq n_0$.
Hence, $\forall x < n_0$, we must have $x \notin S$. Thus, S is bounded below by n_0 and hence, S is a bounded set.
Claim: $\sup S = x_2$.
It is clear that x_2 is an upper bound of S.
Let $\varepsilon > 0$ be an arbitrary positive real number. Since, f has local maximum at x_2, we have $\delta > 0$ such that $f(x) \leq f(x_2)$, for all $x \in (x_2 - \delta, x_2 + \delta)$. Take $x = x_2 - \dfrac{\delta_1}{2}$ where $\delta_1 = \min\{\delta, \varepsilon, x_2 - x_1\}$. Then, as $x_1 < x < x_2$ and f is strictly increasing on (x_1, x_2), we get, $f(x) < f(x_2)$ and hence, we get, $x \in S$ such that $x_2 - \varepsilon < x \leq x_2$.
Thus, by characterization of supremum, $\sup S = x_2$. [4]

Part III

1. Let $\mathbb{Z}_{10}$ denote the set of integers modulo 10.

   (a) i. Find a nonzero solution to the following system of equations in $\mathbb{Z}_{10}$: [2]

   $$4x + 6y = 0$$
   $$2x + 4y = 0$$

   ii. Find a nonzero solution to the following system of equations in $\mathbb{Z}_{10}$: [2]

   $$4x + 3y = 0$$
   $$x + 2y = 0$$

   (b) Prove that the system of equations

   $$ax + by = 0$$
   $$cx + dy = 0$$

   has a unique solution $x = 0, y = 0$ in $\mathbb{Z}_{10}$ if and only if the number $(ad - bc) \pmod{10} \in \{1, 3, 7, 9\}$. [8]

Solution:

(a) i. $x = 5, y = 0$ or any other correct solution [2]
ii. $x = 4, y = 8$ or any other correct solution [2]

(b) Consider the following system of equations where a, b, c, d are integers modulo 10.

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned} \qquad (1)$$

Observe that $x = 0, y = 0$ is a solution of Eq. (1) for all possible values of a, b, c, d.

Claim: The system (1) has a unique solution $x = 0, y = 0$ if and only if $ad - bc$ is a unit modulo 10. (Note that the units modulo 10 are $1, 3, 7, 9$.)Suppose $ad - bc \in \{1, 3, 7, 9\}$ By multiplying the first equation by d and the second by b and then subtracting the one from the other, we get $(ad - bc)x = 0$. As $ad - bc$ is a unit, we then get $x = 0$. Similarly, we have $y = 0$. [2]

For the other implication, we have to make cases. Suppose $ad - bc$ is not a unit modulo 10.

We show that Eq. (1) has a solution $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, i.e., there exists a nontrivial solution.

As $ad - bc$ is not a unit modulo 10, there exists $m \neq 0$ such that $m(ad - bc) = 0$ modulo 10. In fact m can be chosen to be 1, 2 or 5.

First observe that $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} mb \\ -ma \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} md \\ -mc \end{pmatrix}$ are solutions of Eq. (1).

If at least one of the four terms $-ma, mb, -mc, md$ is nonzero modulo 10, we are done! We have to handle the case where $-ma = mb = -mc = md = 0$ modulo 10.

If $m = 1$, i.e. if $ad - bc = 0$ modulo 10, then in the above case, $a = b = c = d = 0$ and thus $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a nontrivial solution of Eq. (1) and we are done.

So we consider the case $ad - bc \neq 0$ modulo 10.

If $m = 2$. then in this case, each of $a, b, c, d \in \{0, 5\}$ (mod 10). Thus $x = 2, y = 0$ is a nontrivial solution.
If $m = 5$, then in this case, each of $a, b, c, d \in \{0, 2, 4, 6, 8\}$ (mod 10). Thus $x = 5, y = 0$ is a nontrivial solution. [6]

2. Let $f(x) = a_0 + a_1x + a_2x^2 + a_{10}x^{10} + a_{11}x^{11} + a_{12}x^{12} + a_{13}x^{13}$ and $g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_{11}x^{11} + b_{12}x^{12} + b_{13}x^{13}$ be polynomials with real coefficients such that $a_{13} \neq 0, b_3 \neq 0$. Prove that the degree of $gcd(f, g) \leq 6$. [12]
Solution: Define $f_1(x) = a_0 + a_1x + a_2x^2, f_2(x) = a_{10} + a_{11}x + a_{12}x^2 + a_{13}x^3$ and $g_1(x) = b_0 + b_1x + b_2x^2 + b_3x^3, g_2(x) = b_{11}x + b_{12}x^2 + b_{13}x^3$. As $a_{13} \neq 0$ and $b_3 \neq 0$, we have degree$(f_2(x)) = 3$ and degree$(g_1(x)) = 3$, so that degree$(f_2(x)g_1(x)) = 6$. Now note that

$$f(x) = f_1(x) + x^{10}f_2(x),$$
$$g(x) = g_1(x) + x^{10}g_2(x).$$

[6]

Observe that $f(x)g_2(x) - g(x)f_2(x) = f_1(x)g_2(x) - g_1(x)f_2(x). \quad \cdots \quad$ [I]
Note that the degree of the polynomial on RHS of [I] is equal to 6. If $h = \gcd(f, g)$, then h divides the polynomial on LHS of [I]. Therefore h divides the polynomial on RHS of [I]. Thus degree of $gcd(f, g) \leq 6$. [6]

3. (a) Let x_0 be an arbitrary real number. Define sequence (x_n) as follows:

$$x_n = \frac{x_{n-1} + 4}{5}, \forall n \geq 1.$$

Show that sequence (x_n) is convergent. Find $\lim_{n\to\infty} x_n$. [6]

(b) Let n be a fixed positive integer. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-zero function satisfying following conditions:

i n^{th} derivative of f is continuous.
ii $f(5x + 3) = 5^n f(x + \frac{7}{5}), \forall x \in \mathbb{R}$.

Show that f is a polynomial of degree n. [7]

Solution:

(a) Let $x_n = \dfrac{x_{n-1} + 4}{5}$. Observe that

$$x_{n+1} - x_n = \frac{x_n - x_{n-1}}{5} = \frac{x_{n-1} - x_{n-2}}{5^2} = \frac{x_1 - x_0}{5^n}.$$

This implies that the sequence (x_n) is Cauchy sequence of real numbers and hence convergent. Suppose the sequence (x_n) converges to c. Then by the given recurrence relation, we have $c = \dfrac{c+4}{5}$. Therefore $c = 1$. [6]

(b) Differentiating $f(5x+3) = 5^n f(x+\frac{7}{5})$ n times, we get that $5^n f^{(n)}(5x+3) = 5^n f^{(n)}(x+\frac{7}{5})$. Therefore $f^{(n)}(5x+3) = f^{(n)}(x+\frac{7}{5})$.
Put $y = 5x+3$, then $x = (y-3)/5$. Therefore for all y,

$$f^{(n)}(y) = f^{(n)}\left(\frac{y-3}{5} + \frac{7}{5}\right) = f^{(n)}\left(\frac{y+4}{5}\right).$$

Put $y = x_0$. Now by using the sequence in part (a), we have $f^{(n)}(x_0) = f^{(n)}\left(\dfrac{x_0+4}{5}\right) = f^{(n)}(x_1)$. Continuing in this way, we get that $f^{(n)}(x_m)$ is a constant sequence. Since the sequence (x_m) converges to 1, by continuity of $f^{(n)}$ we have $f^{(n)}(x_m)$ converges to $f^{(n)}(1)$. Thus $f^{(n)}(x_m) = f^{(n)}(1)$, $\forall m$. Since x_0 is an arbitrary real number, $f^{(n)}(x)$ is a constant function. Hence f is a polynomial of degree at most n. However by comparing the coefficient of highest degree term in the equation $f(5x+3) = 5^n f(x+\frac{7}{5})$, we get that f must be a polynomial of degree n.

[7]

4. Let n be a positive integer bigger than 1. Let $\rho(n)$ be the smallest possible rank of an $n \times n$ matrix that has zeros along the main diagonal and strictly positive real numbers off the main diagonal.

(a) Find $\rho(2)$ and $\rho(3)$. [4]

(b) Find $\rho(4)$. [5]

(c) Find $\rho(n)$ for each n. [4]

Solution:

(a) For $n = 2$, the determinant of such a matrix is negative. Therefore $\rho(2) = 2$. [1]
Let $n = 3$.
Claim: All three rows R_1, R_2, R_3 are linearly independent.
Suppose $c_1R_1 + c_2R_2 + c_3R_3 = 0$. In the first column, first entry is zero and all other entries are positive. Therefore, we get that either c_2, c_3 have opposite signs or both are zero. The same argument applies to the pairs c_1, c_2 and c_1, c_3. Hence they all must be zero. Hence $\rho(3) = 3$. [3]

(b) Let $n = 4$.
By similar argument as in the case of $n = 3$, the first three rows are linearly independent.
Consider a 4×4 matrix $A = ((i-j)^2)_{i,j=1}^4$. Note that rank of A is 3. Hence $\rho(4) = 3$. (Any other correct example). [5]

(c) We show that for all $n \geq 3, \rho(n) = 3$.
By similar argument as in the case of $n = 3$, the first three rows are linearly independent.
We now present an example of a matrix of rank at most 3.
Consider an $n \times n$ matirx $A = ((i-j)^2)_{i,j=1}^n$.

Note that $A = \begin{pmatrix} 1^2 \\ 2^2 \\ \vdots \\ n^2 \end{pmatrix} (1, 1, \cdots, 1) - 2 \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix} (1, 2, \cdots, n) + \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} (1^2, 2^2, \cdots, n^2)$.

Since A is the sum of three matrices of rank 1, the rank of A is at most 3. [4]
(Any other correct example).